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Simple Groups from Order 2001 to Order 3640.

By L. P. SICELOFF.

1. The groups of prime order, the alternating groups, and other infinite systems* of finite simple groups include most, but not all, of the known finite simple groups. All of the simple groups having orders less than 2001 have been determined;† but beyond this very limited range a problem of fundamental importance still presents itself: to determine precisely for what orders simple groups exist, and to determine every type of simple group for each such order.

In the field here covered, from order 2001 to order 3640, it turns out that the only possible orders of simple groups are the well-known cases, viz., there is one and only one‡ simple group of each of the orders 2448, 3420, and there is one of order $2520 = 7! \div 2$. Whether there is more than one of order 2520 has not been determined.

2. In giving a brief account of the methods employed in this paper, it is to be observed at the outset that the procedure is, in general, negative in character: to determine whether there is a simple group of a given order, the initial attack is an attempt to determine whether every group of this order must be composite. This is, of course, due to the fact that the simple group is defined by exclusion: it must not have an invariant subgroup.

In considering whether every group G of given order $g = p^m q^n \dots$ ($p, q, \dots$, distinct primes) must have an invariant subgroup, we turn our attention to the subgroups that are known to exist for every group of that order. And since the groups of orders $p^m, q^n, \dots$, with their subgroups, are the only subgroups of G whose existence is independent of the type of G ,

* For a list, *Bull. Am. Math. Soc.*, 5 (2 ser.), p. 470.

† To order 200 by Hölder, *Math. Annalen*, XL, 55–58; to 660 by Cole, *AM. JOUR. MATH.*, 14, 378–388, and 15, 303–315; to 1092 by Burnside, *Proc. Lon. Math. Soc.*, 26, 333–338; to 2001, by Ling and Miller, *AM. JOUR. MATH.*, 22, 13–26.

‡ Frobenius, *Berliner Sitzgsb.*, 1902, 359; Burnside, “Theory of Groups,” 317–335.

Sylow's theorems are indispensable for this purpose. In connection with these theorems frequent use is made of the important equation* of Maillet,

$$kp + 1 = 1 + x_1 p + x_2 p^2 + \dots + x_m p^m,$$

in which $kp + 1$ is the number of subgroups of order p^m in G , and $x_r p^r$ is the number of these subgroups having in common with a specified one of them precisely p^{m-r} elements. Note, in particular, that if k is not divisible by p , then $x_1 \neq 0$; i. e., some of the subgroups of order p^m have a common subgroup of order p^{m-1} . Of considerable importance also, in this problem, is the well-known isomorphism between a simple group and a group of substitutions on a conjugate set of its subgroups or elements.

Certain other methods and theorems, useful under special conditions, will be explained in connection with their application. In the following section is proven a theorem which considerably simplifies the preliminary reduction of the list of orders in section 4.

3. THEOREM. *A simple group G of order $g = (p + 1)p^\alpha n$, where p is prime and is not a factor of n , can not contain a conjugate set of $p + 1$ subgroups of order p , unless $\alpha = 1$ and n is a factor of $p - 1$.*

For if there are $(p + 1)H_{p^\alpha}$ in G , G is simply isomorphic with a transitive G^{p+1} . Then p^α must divide $(p + 1)!$, and $\alpha = 1$. Now if G , of order $(p + 1)pn$, has $(p + 1)H_p$, each H_p is invariant in an H_{pn} ; and this affects p symbols when G is expressed as a group of substitutions on $p + 1$ symbols.† But the symmetric G^p contains precisely $p(p - 1)$ substitutions that transform a substitution of order p into its own powers. Then pn must divide $p(p - 1)$.

4. We proceed now to examine the list of orders between 2001 and 3640. If the order of a group belongs to any of the following classes of numbers,‡ the group is composite.

- I. Odd composite numbers § less than 40,000.
- II. Numbers with fewer than six prime factors, || with four exceptions.
- III. Even numbers ¶ not divisible by one of the numbers 12, 16, 56.
- IV. Numbers ** p^m , $p^m q^n$, $p^4 q r$, $p^2 q^2 r^2$.

* *Ann. Fac. Sci.*, Toulouse, 9 (1895), D7. See Burnside, "Theory of Groups," 94.

† For several theorems on groups of degree p or $p + 1$, consult the memoir referred to in note (§) p. 361. See also note (§) p. 364.

‡ For a fuller list of such classes, see Easton's "Constructive Group Theory" (Ginn & Co., 1902), 82, 83. See also *Encyclopédie des Sci. Math.*, t. 1, v. 1, fas. 4, p. 590.

§ Burnside, *Proc. Lon. Math. Soc.*, 33, 268.

|| Burnside, "Theory of Groups", 367.

¶ *Ibid.*, 365.

** On $p^m q^n$, see Burnside, *Proc. Lon. Math. Soc.*, 2 (2 ser.), 432; on $p^4 q r$, Frobenius, *Berl. Sitzgsb.*, 1901, 1329; on $p^2 q^2 r^2$, Maillet, *Quar. Jour. P. and Appl. Math.*, 29, 263.

V. Orders which, for some p and every corresponding k , do not divide $(kp + 1)!$, $kp + 1$ being the number of subgroups of order p^m . This is an immediate consequence of the isomorphism between the abstract group and a substitution group of degree $kp + 1$.

All such numbers being excluded, there remain between 2001 and 3640 the following 45 cases requiring special examination :

*2016 = $2^5 3^{27}$	2520 = $2^3 3^{25} \cdot 7$	3120 = $2^4 3 \cdot 5 \cdot 13$
*2040 = $2^3 \cdot 5 \cdot 17$	**2640 = $2^4 3 \cdot 5 \cdot 11$	3168 = $2^5 3^{21} 11$
2080 = $2^5 \cdot 13$	*2688 = $2^7 3 \cdot 7$	*3240 = $2^3 3^4 5$
*2100 = $2^3 \cdot 5^2 7$	*2700 = $2^2 3^3 5^2$	**3276 = $2^2 3^{27} \cdot 13$
**2112 = $2^6 3 \cdot 11$	**2760 = $2^3 3 \cdot 5 \cdot 23$	*3300 = $2^2 3 \cdot 5^2 11$
*2160 = $2^4 3^3 5$	**2772 = $2^2 3^{27} \cdot 11$	**3312 = $2^4 3^{22} 3$
*2184 = $2^3 3 \cdot 7 \cdot 13$	*2800 = $2^4 5^{27}$	3360 = $2^5 \cdot 5 \cdot 7$
**2208 = $2^6 3 \cdot 23$	*2808 = $2^3 3^{31} 3$	3420 one simple
*2240 = $2^6 5 \cdot 7$	2880 = $2^6 3^{25}$	*3432 = $2^3 3 \cdot 11 \cdot 13$
2268 = $2^3 3^{47}$	*2940 = $2^2 3 \cdot 5 \cdot 7^2$	*3480 = $2^3 \cdot 5 \cdot 29$
*2280 = $2^3 3 \cdot 5 \cdot 19$	2976 = $2^5 3 \cdot 31$	*3520 = $2^6 \cdot 5 \cdot 11$
**2376 = $2^3 3^{31} 11$	3024 = $2^4 3^{37}$	*3528 = $2^3 3^{27} 2$
*2400 = $2^5 3 \cdot 5^2$	**3040 = $2^5 \cdot 19$	**3564 = $2^2 3^{41} 11$
2448 one simple	**3060 = $2^2 3^{25} \cdot 17$	*3600 = $2^4 3^{25} 2$
*2464 = $2^5 7 \cdot 11$	*3080 = $2^3 5 \cdot 7 \cdot 11$	*3640 = $2^3 \cdot 7 \cdot 13$

A. *Orders Easily Rejected.*

5. The orders marked with stars are easily shown to belong only to composite groups. In particular, a direct application of the theorem of section 3 to each of the orders marked with a double star removes these 11 numbers from our list of possibilities. Thus, a simple group of order 2112 must contain 12 $H_{11}^{11} \cdot 16$; but this is impossible, since 16 is not a factor of 10.

We turn then to the examination of the 25 orders marked with a single star.

2016. A simple G_{2016} must have 9, 21 or 63 H_{32} ; and if 63, the H_{16} common to $2k + 1$ of them is invariant in a subgroup of order $32(2k + 1)^m$, and has then 21, 9, 7, or fewer conjugates. $7!$ is not divisible by 32. Thus a simple G_{2016} must be expressible as a transitive group of degree 9 or 21.

There are 8, 36 or 288 H_7 . The theorem of section 3 excludes the 8; 36 $H_7 \cdot 8$ contain elements of order 14, and these are odd, both in the G^9 and in the G^{21} . Then there must be 288 H_7 . These furnish $6 \cdot 288$ elements of order 7, and the

number n of elements whose orders* divide $7 \cdot 32$ is $n = 224x > 6 \cdot 288$; whence x can only be 8, and $n = 1792$. This leaves only $1792 - 6 \cdot 288 = 64$ elements whose orders divide 32. But 3 H_{32} with a common H_{16} furnish 64 different elements, and it is easily seen that the remaining subgroups of order 32 must contain elements other than these 64.

2040. A simple group of this order would contain 120 H_{17} and 51 or more H_5 . But these furnish more than 2040 elements.

2100. A simple G_{2100} must have 21 H_{25} . An H_5 is common to 6 or 21 of these; and if to 21, the group is obviously composite. And if 6 H_{25} have a common H_5 , this is invariant in a subgroup of order $6 \cdot 25m = 300$ at least ($m = 1$ is excluded because the H_{150} with its invariant H_5 gives a factor group Γ_{30} , which leads to an invariant H_{25} in H_{150}), and has therefore at most 7 conjugates. But 25 does not divide $7!$.

2160. There must be 10, 16 or 40 H_{27} ; there is, however,† no transitive G_{2160}^{10} . If there are 16 $H_{27 \cdot 5}^{15}$, each has an invariant H_5 ; but this is impossible, since $5 \cdot 27$ is not a divisor of the order $5^3 4 \cdot 3!$ of the normaliser of H_5 in the symmetric G^{15} .

Thus, there are 40 H_{27} , and the H_9 common to 4 (or more) of them has 20 conjugates (or too few), each invariant in H_{108} . But an H_{108} with 4 H_{27} has‡ an invariant H_4 , which, being invariant also in an H_8 , is invariant in H_{216m} , and has $10/m$ conjugates. Clearly $m = 1$; and even this fails, since there is no transitive G_{2160}^{10} .

2184. There are 14 $H_{13 \cdot 12}^{13}$; but these are metacyclic,§ and therefore half-odd.

2240. The H_{32} common to $2k + 1$ of the 35 H_{64} has at most 7 conjugates. But $7! \neq 64m$.

2280. There are 20 $H_{19 \cdot 6}^{19}$. But the metacyclic $H_{19 \cdot 18}^{19}$, in which our $H_{19 \cdot 6}$ is contained, is half-odd; and since 6 is not a factor of 9, our $H_{19 \cdot 6}$ is half-odd.

2400. The H_5 common to $5k + 1$ of the 16 or 96 H_{25} has at most (see method for order 2100) 8 conjugates. But $8! \neq 25m$.

* In a group of order g , the number of elements whose orders divide a factor r of g is a multiple of r . Frobenius, *Berl. Sitzgsb.*, 1895, 984.

† Cole, *Quar. Jour.*, 27, 39. All primitive substitution groups as far as degree 20 have been determined. In most cases where the results of these determinations are applicable to the work of this paper, however, independent proofs are presented, chiefly for the sake of whatever light such independent procedure may throw on the problem here studied.

‡ See p. 63 of M. O. Tripp's dissertation on groups of order $p^3 q^2$ (Columbia University, 1909).

§ The only transitive substitution groups of prime degree p and order less than p^2 are metacyclic. Every transitive group of degree p is either metacyclic or doubly transitive. See Burnside, *Quar. Jour.*, 37, 215.

2464. The greatest subgroup, of order either 16 or 8, common to 7, 11 or 77 of the $(2k+1) H_{32}$ has 22, 14, 11, or too few, conjugates. But there are 56 $H_{11.4}$, with elements of order 22; and these are either odd or non-constructible when G is expressed as transitive in 22, 14 or 11 symbols.

2688. If some of the 21 H_{128} have a common H_{64} , this has at most 7 conjugates; but $7! \neq 128m$. Then the greatest subgroup common to $2k+1$ of the 21 H_{128} is of order 32, and has 14 conjugates ($k=1$ being the only case that calls for examination). But no simply transitive G^{14} exists, of any order whatsoever.*

2700. The H_5 common to 6 or more of the 36 H_{25} has at most 9 conjugates. But $9! \neq 25m$.

2800. A common H_5 has at most 7 conjugates.

2808. There are 27 $H_{13.8}$, in each of which any H_8 transforms H_{13} into itself. But since $i^8 \equiv 1 \pmod{13}$ has only the four solutions of $i^4 \equiv 1 \pmod{13}$, viz., $i = \pm 1, \pm 5$, some two elements, s, t , of H_8 transform any element u of H_{13} into the same power, u^i . Then st^{-1} is permutable with u , and $H_{13.8}$ contains elements of order 26; these will appear as odd substitutions when G is expressed as a transitive G^{27} .

2940. An H_7 is common to all of the 15 H_{49} . Then G is composite.

3080. There are 22 $H_{7.20}$; each has one H_5 . Then G has 11 (or 1) H_5 , each invariant in $H_{5.56}^{10}$; but this can not be, since $5 \cdot 56$ does not divide the order $5^2 \cdot 2 \cdot 4$, of the normaliser of H_5 in the symmetric G^{10} .

3240. The number of H_{81} is 10 or 40; and if 40, the H_{27} common to 4 (or 10 or 40) of them has 10 (or 4 or 1) conjugates. Then a simple G_{3240} can be expressed as a transitive G^{10} . But there is no † transitive G_{3240}^{10} .

3300. By the theorem of section 3 there can not be 12 H_{11} . Then there must be 100 $H_{11.3}$; they furnish 3000 elements of orders 11, 33. But then the number of elements of odd order in G is $3 \cdot 5^2 \cdot 11k > 3000$, whence $k=4$; i. e., all the elements of G are of odd order. This is impossible.

3360. The theorem of section 3 excludes 8 H_7 . If there are 15 H_7 , there are substitutions of order 14, odd in the 15 symbols; then there are 120 $H_{7.4}$, and these furnish 720 elements of order 7 and at least 720 of order 14.

There are 16, 21, 56, 96 or 336 H_5 . 16 $H_{7.30}$ yield only 1 or 8 H_7 . The case for 21 H_5 fails because the substitutions of 14 would be odd in the G^{21} . The fifth power of a substitution of order 15 in one of the 56 $H_{5.12}$ is a substi-

* Miller, *Quar. Jour.*, 29, 242.

† Cole, *Quar. Jour.*, 27, 39.

tution of order 3, omitting $n = 11, 26$, or 41 of the 56 symbols; and H_3 , with 1 or 4 conjugates in the H_{60} , has in G $56m/n$ ($m = 1$ or 4) conjugates, which is impossible. Again, 96 $H_{5 \cdot 7}$ yield only 1 or 8 H_7 . Thus a simple G_{3360} must have 336 H_5 , giving 1344 elements of order 5. But then the number of elements whose orders divide $5 \cdot 7 \cdot 32$ is $35 \cdot 32x > 1344 + 1440$; whence $x = 3$, and the group has no elements of order 3. This is impossible.

3432. Each of the 78 $H_{11 \cdot 4}$ has an invariant H_2 , which therefore omits 34 (s_{22} must be an even substitution, and has then 2 cycles of 22 symbols with 3 cycles of 11) of the 78 symbols, and has $78k/34$ conjugates. But this is impossible, since k , the number of conjugates which H_2 has, under G , in H_{44} , must be either 3 or of the form $11m + 1$ ($m < 3$).

3480. There are 30 $H_{29 \cdot 4}$; and these, being metacyclic, are half-odd, since 4 does not divide 14.

3520. There are 320 H_{11} . Then the number of elements whose orders divide $11 \cdot 64$ is $11 \cdot 64x > 3200$; whence $x = 5$, and there are no elements of order 5. This can not be.*

3528. The H_7 common to 8 or more of the 36 H_{49} has at most † 9 conjugates. But $9! \neq 49m$.

3600. The H_5 common to 6 or more of the 16 or 36 H_{25} is invariant in a subgroup of order 300 or more; and if more than 300, the H_5 has at most 9 conjugates. But $9! \neq 25m$. If the H_5 is invariant in H_{300} , it has 12 conjugates. But there is no transitive G_{3600} on 12 symbols ‡

3640. The theorem of section 3 excludes 14 H_{13} . Then there must be 40 $H_{13 \cdot 7}$, and each has an invariant H_7 . The total number of H_7 in G can only be 1 or 8. But $8! \neq 13m$.

B. Orders Requiring more Detailed Examination.

6. Apart from the orders 2448, 2520 and 3420, for each of which there is a simple group, there remain in our list seven orders to be examined. That none of these numbers is the order of a simple group appears as follows.

2080. If there are 16 $H_{5 \cdot 26}$, they contain elements of order 65; but these are not expressible in 15 symbols. 416 H_5 would furnish, with 40 H_{13} , too many

*Attention is called to the fact that this and a few others of our orders could have been rejected by an important theorem of Frobenius: Every transitive G^n of class $n - 1$ has a characteristic H_n . *Berl. Sitzgsb.*, 1901, 1226.

† In Maillet's great memoir referred to in note (*), p. 362, above, is proven the following theorem, which applies to orders 2100, 2940, 3528. If, in Maillet's equation (section 2), for every $i > 1$ $x_i = 0$, all the H_{pm} have a common H_{pm-1} , and the group is simple. See Vol. 10, A 5, of the *Annales*.

‡ Miller, *Quar. Jour.*, 28, 219.

elements. Then there must be 26 $H_5 \cdot 16$. Each H_{80} has 5 H_{16} , since only one H_{16} in H_{80} , being invariant also in H_{32} , would be invariant in H_{160m} , and would have 13 conjugates ($m = 1$); the 13 H_{160} contain each an H_{80} with 1 H_5 ; then each H_{160} has an invariant H_5 , and G has not its necessary 26 H_5 . These 5 H_{16} have in common an H_8 or only an H_4 .

If the 5 H_{16} have a common H_8 , they furnish 48 different elements of order 2^i . But observe the 40 $H_{13} \cdot 4$: since an s_{26} would be odd in the transitive G^{26} , each H_{52} has 13 cyclical H_4 , with only identity in common. Each H_4 effects 38 symbols in G^{40} , since s_4 must be even and its square must not transform s_{13} into itself. Then H_4 has in G $40 \cdot 13/2 = 260$ conjugates. But in the G^{26} , s_4 affects 24 symbols, since its square must transform s_{13} into s_{13}^{-1} . Let k be the number of these 260 H_4 in each H_{80} . $26k/2 = 260$, $k = 20$. These 20 cyclical H_4 furnish 40 s_4 and 10 s_2 in each H_{80} . But this can not be, since the 5 H_{16} with a common H_8 yield only 48 elements of order 2^i .

Then the 5 H_{16} have only a common H_4 . Let J denote this H_4 , and a, b, c , its elements of order 2 (the H_4 can not be cyclical, since there would then be substitutions of order 20, odd in the G^{26}). a, b, c , being permutable with the elements of H_5 , affect 20 symbols in the G^{26} . Then a has $26m/6$ conjugates in G , m being the number of conjugates which a has, under G , in H_{80} . Observe that m must be a multiple of 3. Now a can not be conjugate with an s_2 of the 20 cyclical H_4 in H_{80} , since such an s_2 affects 24 symbols. But these 20 H_4 , with J , furnish 53 of the 63 elements of order 2^i in H_{80} . Then we must find the conjugates of a among the remaining 10 elements of the 5 H_{16} . These 10 elements must form, under H_{80} , two conjugate sets of 5 each, or else one set of 10. Thus, a may have in H_{80} , $m = 5n + (1, 2, \text{ or } 3) = 3, 6, \text{ or } 12$ conjugates. If $m = 3$, G is a doubly transitive* G^{13} . This can not be, for 12 does not divide 2080. If $m = 6$ or 12, then either b or c has only one conjugate in H_{80} , and $26/6$ conjugates in G . This is impossible.

2268. In the equation

$$28 = 1 + 3x + 9y + 27z,$$

if $x \neq 0$, an H_{27} common to 4, 7, or 28 H_{81} has at most 7 conjugates; but $7! \neq 81m$. Therefore $x = 0$, and the only solutions for the equation are (a) $x = z = 0$, $y = 3$, and (b) $x = y = 0$, $z = 1$.

Under the solution (b), each H_{81} has in common with each other H_{81} precisely 3 elements. A common H_8 belongs in 4, 7, or 28 H_{81} . Thus each H_{81} has

* See foot-note (§), p. 364.

at least 62 elements not in any other H_{81} ; and the 28 H_{81} furnish more than $28 \cdot 62 = 1736$ elements. But then the number of elements whose orders divide $7 \cdot 81$ is $567j > 1736$, whence $j = 4$, and there are no elements of order 2. This is impossible.

Under solution (a), each H_{81} has an H_9 in common with each other H_{81} . Denote by N such an H_9 . If N is invariant in an H_{81} , it is invariant in an H_{81k} , with $k \geq 4$, and has at most 7 conjugates. This can not be. Thus N is invariant in only an H_{27} from each of the 4 or 7 H_{81} in which it lies. If N is common to 7 H_{81} , the 12 $H_{27} \cdot 7$ yield only 1 H_7 . Therefore N is common to precisely 4 H_{81} , is invariant in an $H_{27} \cdot 4g$, and has $21/g$ conjugates. Obviously, $g = 1$, N has 21 conjugates, each invariant in an H_{108} . If N has, under G , n conjugates in each H_{81} , $28n/4 = 21$, $n = 3$. Moreover, a single H_{27} of an H_{81} contains the 3 conjugates of N that lie in that H_{81} ; for an H_{27} , being invariant in an H_{81} , has 28 conjugates, and $28h/4 = 21$ gives $h = 3$. Denote by P, Q the other two conjugates of N in $T \equiv H_{27}$. N is common to the first H_{81} and three other H_{81} ; P is common to the first H_{81} and a second set of three others; and similarly for Q . Now if T is abelian, the H_3 common to N and P is invariant in 2 H_{108} , and has 12, 7, or fewer conjugates. These possibilities have already been rejected. Therefore T is not abelian; it has 4 H_9 with a common H_3 which, being the only H_3 invariant in T , is the central H_3 of the first H_{81} .

The arrangement outlined above exhibits the first H_{81} as belonging to three sets of 4 H_{81} , each set having a common H_9 . This arrangement involves only 10 H_{81} , and must be accompanied by two other such arrangements, in order that the first H_{81} may appear as having an H_9 in common with *every* H_{81} , as required in the case (a) under discussion. It follows that the first H_{81} has 9 H_9 lying in sets of three in three non-abelian H_{27} . But observe that any two of these 3 H_{27} have a common H_9 , which contains the central H_3 of each. Both of these centrals are invariant in H_{81} , and therefore coincide, since the H_9 common to the 2 H_{27} has only 1 H_3 that is invariant in either H_{27} . Then this central H_3 is common to all the 9 H_9 , and is thus common to all the 28 H_{81} . Similarly, the central of each H_{81} must be common to all the 28 H_{81} . But since 2 H_{81} must not have more than an H_9 in common, the central of one H_{81} must coincide with the central of another H_{81} ; and this central is invariant in an H_{81r} , where $r \geq 4$. Then it has at most 7 conjugates, which is impossible.

2880. There are 16, 36, 96, or 576 H_6 . And there can not be 16 $H_6 \cdot 36$, since $5 \cdot 36$ does not divide the order $5^3 4 \cdot 3!$ of the normaliser of H_6 in the symmetric G^{16} .

If there are 96 $H_{6 \cdot 5}$, an s_3 permutable with an s_5 omits $k = 6, 21, 36, \dots$, of the 96 symbols, and has $96/k$ conjugates. Then $k = 6$; $\{s_2\}$ has 16 conjugates, each invariant in an H_{180} . If the corresponding factor-group Γ_{60} has only 1 H_5 , H_{180} has only 1 H_6 , and G has fewer than 96 H_5 . Thus, Γ_{60} must have 6 H_5 , and H_{180} has 6 $H_{15 \cdot 2}$. Then each H_5 affects 15 symbols in the transitive G^{16} , and s_{15} is a circular substitution. s_{15} is not permutable with s_2 in H_{30} , since the resulting s_{30} would not be expressible in the 15 symbols. Therefore s_2 consists of seven transpositions and is odd, since it transforms s_{15} into s_{15}^{-1} .

There can not be 576 H_5 , since the transitive G^{576} would be of class 575, and would therefore contain * a characteristic H_{576} .

Thus, if there is a simple G_{2880} , it must have 36 H_5 , each invariant in an H_{80} . Each H_{80} has 5 H_{16} ; for if it had only 1 H_{16} , this would be invariant in an $H_{5 \cdot 32m}$ containing our H_{80} , and the $36/m$ conjugates of the H_{160m} would not yield 36 H_5 . Now the greatest subgroup S common to the 5 H_{16} of H_{80} is of order 4 or 8. S contains a substitution s_2 permutable with each s_5 of H_5 ; this s_2 affects 20 of the 36 symbols, and has then $36n/16$ conjugates. To determine n , consider any element t of order 2 in the H_{80} but not in S ; t must transform each s_5 into s_5^{-1} . Then t affects 28 symbols at least, and can not be conjugate with our s_2 . Thus S contains all the conjugates of our s_2 which are in the H_{80} , when the s_2 is transformed by all the elements of G . Therefore there is in S an s_2 for which $n = 1, 3, 5$, or 7 , and the number, $36n/16$, of its conjugates in G is not an integer. This is impossible.

2976. The 32 $H_{31 \cdot 3}$ obviously can not be cyclic. Then in our transitive G^{32} , each s_3 affects 30 symbols, and G has $32 \cdot 31/2 = 496$ H_3 .

There are 31 or 93 H_{32} . But a transitive G^{31} must be doubly transitive or metacyclic, both of which are impossible here. Thus there are 93 H_{32} ; and if 3 (or 31) of these have a common H_{16} , this has 31 (or 3 or 1) conjugates. This can not be, since there is no transitive G_{2976}^{31} . Then there must be an H_8 common to 3 (or 31) of the 93 H_{32} . This H_8 is invariant in an $H_{3 \cdot 16m}$; and if $m > 1$, the H_8 has 31, 3, or 1 conjugate. Thus, $m = 1$; there are 62 H_{48} , and since G has 496 H_3 , each H_{48} has 16 H_3 and therefore only 1 H_{16} , which, being invariant also in an H_{32} , is invariant in an H_{96k} and has 1 or 31 conjugates. Every G_{2976} is therefore composite.

3024. In a simple G_{3024} there must be 36 $H_{7 \cdot 12}$. In one of these an s_3 can not be permutable with an s_7 , since $\{s_3\}$, affecting 21 symbols in G^{36} , would then have $36k/15$ conjugates, where $k = 1$ or 4 .

The solutions of $i^4 \equiv 1 \pmod{7}$ are also the solutions of $i^2 \equiv 1 \pmod{7}$, viz., ± 1 . Therefore any H_4 in an H_{84} has an s_2 permutable with every element of the H_7 , and the elements permutable with H_7 form a group S of order 14 or 28 in H_{84} . S is invariant in H_{84} .

If S is of order 28, it has an H_4 which is invariant in H_{84} and also in an H_8 . H_4 is then invariant in an H_{168m} . If $m > 2$, H_4 has too few conjugates. And if $m = 1, 2$, then H_{168m} has only 1 H_7 , so that G has fewer than its required 36 H_7 .

Thus, S must be of order 14. The subgroup T of order 2 in S is invariant in H_{84} . T affects 28 of the 36 symbols. T has, under G , only one conjugate in H_{84} , since any other H_2 in H_{84} transforms an s_7 into its inverse and therefore, affecting at least 30 symbols, can not be conjugate with T . Then S has in G 36/8 conjugates, which is impossible.

3120. There must be 16, 26 or 156 H_5 . 16 H_5 .₃₉ would yield only 1 H_{13} . If there are 26 H_5 , each is invariant in H_{120} , in which an s_3 permutable with an s_5 affects 15 of the 26 symbols; and $\{s_3\}$ therefore has $26k/11$ conjugates, with $k = 1$ or 4 since an H_{120} with an invariant H_5 has 1 or 4 H_3 . This can not be. Thus there can only be 156 H_5 .

There are 40 H_{13} .₆. An H_{78} contains elements of one or more of the orders 6, 26, 39. An s_{26} would be odd in our G^{40} , and if there is an s_{39} in H_{78} , then H_{78} has an s_2 which consists of 19 transpositions and is odd, since it transforms the circular s_{39} into its inverse. Thus H_{78} has an s_6 . This affects 36 or 39 of the 40 symbols, and generates a cyclical H_6 with $40 \cdot 13/(1, 4) = 520$ or 130 conjugates. The case for 520 cyclical H_6 fails, since these, with the 40 H_{13} and the 156 H_5 , would furnish too many elements.

Therefore, there can only be 130 cyclical H_6 , 130 H_3 , 130 H_2 , each invariant in an H_{24} . Each H_3 in H_{24} affects an even number of the 130 symbols and therefore occurs in 2 of the 130 H_{24} . If each H_{24} has only 1 H_8 , this is invariant in 2 H_{24} and therefore in a group S which contains several H_3 , each invariant in an H_{24} in S . Then the order of S is 240 or more; and if more than 240, S has too few conjugates; while if S is of order 240, its 13 conjugates yield fewer than the required 156 H_5 in G . It follows that each H_{24} must have 3 H_3 ; these have a common H_4 invariant in H_{24} . An H_8 of H_{24} , being in 2 of the 130 H_{24} , contributes an invariant H_4 to each. If these 2 H_4 are the same H_4 , this is invariant in a group S as described and rejected above. And again, if the 2 H_4 are different, their common H_2 , being invariant under an H_8 and an H_3 in each of the 2 H_{24} , is invariant in each of the 2 H_{24} and therefore in a group S as described and rejected above. Every G_{3120} is composite.

3168. A simple G_{3168} must have 12 or 144 H_{11} . The theorem of section 3 excludes the 12.

There can not be 16 H_9 , for 16 H_9 's yield only 1 H_{11} in G . Then there are 22, 88, or 352 H_9 ; and if 22 or 88, two of these have a common H_3 . Moreover, if no two of 352 H_9 had a common H_3 , these with 144 H_{11} would furnish too many elements. Hence in all cases a subgroup L of order 3 is common to $3k + 1 = 4, 16, 22$ or more H_9 .

If L is common to more than 22 H_9 , it has too few conjugates; and if to 22 H_9 , the 16 $H_{22 \cdot 9k}$ ($k = 1$) yield only 1 H_{11} .

If L is common to 16 H_9 , it is invariant in an $H_{16 \cdot 9k}$, $k = 1$. The factor group Γ_{48} has 16 H_3 and therefore 1 H_{16} , whence H_{144} has an invariant H_{48} . If this H_{48} has an invariant H_{16} , it is invariant in H_{144} and also in an H_{32} , and has too few conjugates. Hence the H_{48} has an invariant H_8 , which is invariant in H_{144} . $H_{144}/H_8 = \Gamma_{18}$ has 1 H_9 ; then H_{144} has an H_{72} and only 4 H_9 . This can not be.

Thus L is common to only 4 H_9 , and is invariant in an $H_{9 \cdot 4k}$, $k = 1, 2$, or 4.

(a) $k = 1$. Each of the 88 H_{36} has an invariant H_4 , which is invariant in H_{72m} , $m = 1, 2$. If $m = 1$, L is invariant in H_{72} ; this can not be. If $m = 2$, $H_{144}/H_4 = \Gamma_{36}$ has an invariant H_9 or H_4 ; H_{144} has an invariant H_{36} or H_{16} . In the former case, L is invariant in H_{144} ; in the latter, H_{16} is invariant in H_{288} . Both are impossible.

Observe that the case for 352 H_9 is here disposed of, since for them $k = 1$.

(b) $k = 2$. L has 44 conjugates, each invariant in an H_{72} . If L has m conjugates in each H_9 , then for 22 H_9 $22m/4 = 44$, $m = 8$; this is impossible. There must be 88 H_9 . $88m/4 = 44$, $m = 2$; each H_9 has 2 conjugates of L . H_{72} has 5 of them, and they affect 39 symbols in G^{44} .

If H_{72} has only 1 H_8 , this is invariant in H_{144} , which contains L invariantly. This can not be. Hence H_{72} has 3 or 9 H_8 with* a common quadratic H_4 which is invariant in H_{72} . An H_8 in H_{72} occurs in 2 H_{72} , and contributes to each an invariant H_4 . If these 2 H_4 are the same H_4 , this is invariant in 2 H_{72} , and therefore in H_{144m} , $m = 1$; but then L is invariant in H_{144} , which is impossible. Hence the 2 H_4 are different; they contain 5 H_2 , and the H_8 is dihedral or $(1, 1, 1)$. The element s of order 2 common to the 2 H_4 is invariant in an H_{24} from each of the 2 H_{72} , and therefore in a subgroup J which has several H_3 , each invariant in an H_{24} of J . J must be of order 96; s has 33 conjugates in G .

Let t denote an s_3 of L . Denote by u an s_3 of L 's conjugate in an H_9

* See p. 57 of *loc. cit.* in note (§), p. 364.

of H_{72} . t is not invariant in H_{72} . For if it were, u would be conjugate (under H_{72}) only with its first power. Then in the H_{18} of H_{72} under which the chosen H_9 is invariant, an H_2 would be invariant under H_9 and an H_4 , and therefore under H_{36} or H_{72} . In both cases H_{72} would have only 1 H_9 . This is impossible. It follows that t is invariant in H_{36} of H_{72} .

If the H_8 in H_{72} are dihedral, there are nine of them. For an H_2 of the invariant H_4 is invariant in an H_{24} containing L , and L does not transform H_8 into itself; H_{24} has 3 H_8 . But a dihedral H_8 does not transform into itself a second H_2 of the invariant H_4 . Hence, the 3 H_{24} furnish 9 H_8 . These yield $9 \cdot 2 + 3 = 21$ (or more) H_2 conjugate under G . The 44 H_{72} give $44 \cdot 21/m = 33$, $m = 28$; each H_2 of the set omits 28 symbols. But there is in H_{72} an s_3 , affecting 39 symbols, which is transformed into its square by such an H_2 , and this H_2 can not omit more than 18 symbols.

Hence the H_8 are (1, 1, 1). But there is* no H_{72} with 4 non-cyclic H_9 and 3 abelian H_8 .

(c) $k = 4$. L has 22 conjugates, each invariant in H_{144} with 4 H_9 . If there are 22 H_9 in G , $22m/4 = 22$, $m = 4$; L has 4 conjugates in each H_9 . H_{144} has 13 of these 22 H_8 , and L affects 9 symbols. But the 3 H_8 other than L in an $H_9 \equiv N$ of H_{144} affect the other 9 H_8 , other than L , in H_{144} . Then N affects only these 9 symbols. But $22 \cdot 4/n = 22$, $n = 4$; each H_9 must affect 18 symbols.

Thus there are 88 H_9 in G . L has only one conjugate in each H_9 , and affects 21 symbols. If t ($=$ an s_3 of L) is invariant in H_{144} , this has 3 H_{16} ; each of these is invariant in H_{96m} ; $m \neq 1$, for L is not invariant in H_{96} ; and clearly m is not > 1 . Hence, t is invariant in only H_{72} of H_{144} . Every H_2 of this H_{72} , being even and permutable with t , affects just 12 symbols. Hence each H_2 of our H_{72} occurs in 10 H_{144} . Moreover, the H_2 is permutable with the elements of the 10 conjugates of L in the 10 H_{144} , since an s_2 which transforms into its square an s_3 of a conjugate of L affects at least 14 symbols. Thus, the H_2 under observation is invariant in a subgroup which contains precisely 10 conjugates of L . No such subgroup can be constructed.

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* See p. 60 of *loc. cit.* in note (†), p. 364.